STRUCTURES ON FOLIATIONS

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Declaration

Most of Theorem 1 and Corollary 4 of Chapter I were used in my dissertation submitted for the M.Sc. degree in1973.

Summary

In this thesis we consider various structures on foliations. In Chapter I we look at PL and topological foliations and note that not every topological foliation can be made PL. We show that every proper leaf has a microbundle normal to the foliation with holonomy structure group. For transverse foliations the fibres can be chosen not only normal to the leaves of the foliation containing the base leaf, but contained in the leaves of the other foliation. Thus normal microbundles are unique up to isotopy. We also look into the relationship between the holonomy group and the foliated neighborhood of a leaf.

In Chapter II we study differentiable structures on foliations, showing that differentiability conditions are meaningful in a topological sense. We do this by constructing an example of a C^r foliation which is not homeomorphic to any C^{r+1} foliation, $r \geq 0$. (The example is the suspension of a diffeomorphism of a two-manifold.) Using results from Chapter I we also show that the foliation is not C^s integrably homotopic to any C^{r+1} foliation, $0 \leqslant s \leqslant r$.

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Chapter I.

Structure of a Foliated Neighborhood

Introduction

C. Ehresman (2) has shown that if a leaf L of a smooth foliation has a foliated neighborhood, then there exists a fibre bundle over L, normal to the leaves, with discrete structure group. Using the concept of a microbundle and the n-isotopy extension theorem we find a similar result for both PL and TOP catagories, and, in addition, show that the structure group can be chosen to be the holonomy group of L. We use the fact that the constructed normal microbundle can be chosen to have its fibres contained in the leaves of a transverse foliation to demonstrate isotopy uniqueness of normal microbundles. As for applications we show that holonomy characterizes the foliated neighborhood of a leaf (proved by Haefliger in the differentiable case (3)). In particular, if the holonomy group of a compact leaf L is trivial then the leaf has a trivial foliated neighborhood, and if it is finite it has a neighborhood of compact leaves which are covering spaces of L. Another corollary is the known result that a proper submersion is a fibration.

It has been pointed out that some of the results of this chapter are contained in the last part of Siebenmann (7). Since his proofs appear to depend on the complicated machine which is the main context of (7), some justification is felt in presenting the short, direct proofs contained here.

The work is done in both TOP and PL categories unless otherwise specified.

§1. Some Definitions and Examples

1.1 Foliations.

Let I denote the open unit interval between -1 and +1.

Definition. A codimension q foliation $\mathcal{F}$ of an n-manifold M, denoted $(M,\mathcal{F})$, consists of a collection of disjoint leaves $\{ L_\alpha : \alpha \in A \}$ such that $M = \underset{\alpha}{\cup} L_\alpha$. Furthermore we require: If x is in L_α there exists a neighborhood U of x in M, a neighborhood G of x in L homeomorphic to I^{n-q}, and a homeomorphism $g: G \times I^q \overset{\cong}{\to} U$, denoted (g,G), such that $g(G\times 0) = G$ and $g(G\times\text{point})$ is a component of $U \cap L_\beta$ for some β in A.

The set (g,G) is called a distinguished chart of $\mathcal{F}$ which will be confused with the set U.

We make two observations concerning foliations in the PL category. First, if $(M,\mathcal{F})$ is PL, then M can be triangulated such that the chart maps and overlap maps are simplicial, considering the distinguished charts as local charts in M. Hence the leaves are parallel within simplexes.

Secondly, if $(M,\mathcal{F})$ is smooth, it is not always possible to approximate $\mathcal{F}$ by a similar PL foliation within some triangulation of M. For example, there is no PL homeomorphism of the circle with a countable set of discrete fixed points. However, there are such smooth homeomorphisms and the suspension of any of them gives a smooth foliation which is not homeomorphic to any PL foliation.

1.2 Geometric Interpretation of Holonomy.

Haefliger gives a detailed discussion of holonomy in (4). In this section we recall the relevant part of his geometric interpretation.

Let $\mathcal{F}$ be a codimension q foliation of an n-manifold M. Let C be a path in a leaf L and let T_o and T_1 be q-discs transverse to $\mathcal{F}$ containing $z_o = C(0)$ and $z_1 = C(1)$, respectively. Then for every neighborhood U of C in M there is a homeomorphism Φ_c from a neighborhood of z_o in T_o onto a neighborhood of z_1 in T_1 satisfying the properties:

(i) If Φ_c is defined on $z \in T_o$, then $\Phi_c(z)$ is contained in the intersection of T_1 and the leaf passing through z in the foliation induced on U by $\mathcal{F}$.

(ii) The germ of Φ_c at z_o is independent of U and of the path C within its homotopy class γ.

Suppose $z_o = z_1$ and $T_o = T_1$. Let f' be the restriction to $f^{-1}(T_o)$ of a distinguished map f such that $f^{-1}(z_o) = (z_o, 0)$.

Then define the holonomy group of L to be the image of the representation $\Phi: \pi_1(L, z_o) \to PL(q)$, defined in a neighborhood of 0, where γ is mapped to $f'^{-1}\Phi_c f'$ and PL(q) is the discrete group of germs of PL homeomorphisms of R^q, defined in a neighborhood of 0, and keeping 0 fixed. The representation $\Phi: \pi_1(L, z_o) \to PL(q)$ is called the holonomy of L.

To actually construct Φ_c, consider a sequence of distinguished charts (f_i, V_i), $i = 1, \ldots, r$, and an increasing sequence t_i of points on $[0,1]$, $t_o = 0$ and $t_r = 1$, such that $C(\,t_k, t_{k+1})$

$\subset V_k$. Let T^i be q-discs transverse to $\mathcal{F}$ containing $C(t_i)$ and such that $T^0 = T_0$ and $T^r = T_1$. For each $i < r$, there is a homeomorphism Φ_i from a neighborhood of $C(t_i)$ in T^i onto a neighborhood of $C(t_{i+1})$ in T^{i+1} such that, if $z_{i+1} = \Phi_i(z_i)$, z_{i+1} is in the leaf passing through z_i in $V_i \cap U$. Then Φ_c is the composition of the homeomorphisms $\Phi_{r-1}\Phi_{r-2}\cdots\Phi_0$.

For further details see Haefliger (4).

1.3 Microbundles with Discrete Group.

Definition. A q-microbundle is a diagram $B \xrightarrow{i} E \xrightarrow{p} B$ where B and E are manifolds, i and p are continuous maps called the injection and projection maps respectively. The composition $p \circ i$ is required to be the identity. Furthermore we require:

Local triviality condition. For every x in B there should exist an open neighborhood G_x of x in B, an open neighborhood U_x of $i(x)$ in E, and a homeomorphism $g_x : G_x \times I^q \to U_x$, defined in a neighborhood of 0, such that the diagram

$$\begin{array}{ccc} G_x & \subset & B \\ \times 0 \cap & & \cap i \\ G_x \times I^q \xrightarrow{g_x} U_x & \subset & E \\ \pi_1 \downarrow & & \downarrow p \\ G_x & \subset & B \end{array}$$

commutes. The diagram is called a chart near x and is denoted (g_x, G_x). An atlas is a collection of charts covering B.

Two microbundles $B \to E_j \xrightarrow{p_j} B$, $j = 1,2$, are isomorphic if there is a homeomorphism $f: E_1 \to E_2$ which is defined on some neighborhood of $i_1(B)$ and has for image some neighborhood of $i_2(B)$ and such that the diagram

$$\begin{tikzcd} B \arrow[r, phantom, "="] \arrow[d, "i_1"'] & B \arrow[d, "i_2"] \\ E_1 \arrow[r, "f"] \arrow[d, "p_1"'] & E_2 \arrow[d, "p_2"] \\ B \arrow[r, phantom, "="] & B \end{tikzcd}$$

commutes, where it is defined.

If (g_1, G_1) and (g_2, G_2) are two charts in a microbundle E we can define a comparison map $h\colon (G_1 \cap G_2) \times I^q \to (G_1 \cap G_2) \times I^q$ by $h = g_2^{-1} \cdot g_1$ where defined.

Observe that (i) $h|(G_1 \cap G_2) \times 0 =$ identity and

(ii) h is fibre preserving.

Then define a function $h'\colon G_1 \cap G_2 \to PL(q)$ by

$$h'(x)y = \pi_2 h(x,y).$$

We say that E has discrete structure group if h' is locally constant.

§ 2. Microbundles over a Leaf.

Let M be an n-manifold with codimension q foliation $\mathcal{F}$. We are interested in constructing a projection defined in a neighborhood of a leaf L, running transversally across leaves onto L itself. In Theorem 1 we will examine the simpler case where L is proper so that L has a foliated neighborhood E embedded in M. In general, consider the induced neighborhood of L.

Suppose there exist injective and projective maps i and p such that the diagram $L \xrightarrow{i} E \xrightarrow{p} L$ is a q-microbundle over L. Then the microbundle E is normal to the foliation $\mathcal{F}$ if the charts in E are also distinguished charts in $\mathcal{F}$.

In all cases we will be concerned with microbundles normal to a foliation so that the microbundle fibres run transverse to the leaves of the foliation.

2.1 Existence of normal microbundles.

The results depend upon the following q-isotopy extension theorem.

Theorem. If $i: N\times I^q \to M\times I^q$ is an allowable q-isotopy, fixed on $i_0^{-1}(M)$, N compact, and if i is locally unknotted in the PL case or locally trivial in the TOP case, then there exists an ambient q-isotopy of M, fixed on M and extending i.

Proof. For the PL case see Hudson (5). For the topological case see Davis (1).

Definition. If (g_i, G_i), i=1,2, are charts of a foliation $\mathcal{F}$, then $g_1: G_1\times I^q \to M$ matches $g_2: G_2\times I^q \to M$ if the verticals match, that is, if $\pi_1 g_2^{-1} g_1(x\times I^q) = x$ where defined. We say that g_1 level preserves with g_2 if $\pi_2 g_2^{-1} g_1(G_1 \cap G_2\times t) = t$ where defined.

Theorem 1.

Let M be a manifold with a codimension q foliation $\mathcal{F}$. A proper leaf L of $\mathcal{F}$ has a microbundle normal to $\mathcal{F}$ with discrete structure group equal to the holonomy group of L.

Proof. Since L is proper and connected we can choose distinguished charts $f_i\colon A_i \times I^q \to M$ such that $\overline{f_i(A_i \times I^q)}$ meets L only in $\overline{A}_i$, and an open cover $\{F_i\}$ of L such that $\overline{F}_i \subset A_i \subset L$, with $\bigcup_{i=1}^{n} F_i$ connected for all n.

Using these charts we will inductively build a microbundle over L.

Induction statement. For each $k \geq 1$ there is an open neighborhood B_k of $\bigcup_{i=1}^{k} \overline{F}_i$ in L and a normal microbundle

$$B_k \xrightarrow{i_k} E_k \xrightarrow{p_k} B_k$$

which extends $E_{k-1}/\bigcup_{i=1}^{k-1} F_i$ such that $\overline{E}_k \cap \overline{L} = \overline{B}_k$. Furthermore, E_k has discrete structure group contained in the holonomy group of L, $\mathcal{H}(L)$.

For $k = 1$ let $B_1 = A_1$. Then E_1 is simply $f_1(B_1 \times I^q)$ with the projection $p_1 = \pi_1 f_1^{-1}|_{E_1}$.

Assuming the induction statement is true for k=n, we show that it holds for k=n+1. Thus we have an open neighborhood B_n of $\bigcup_{i=1}^{n} \overline{F}_i$, a normal microbundle $B_n \to E_n \xrightarrow{p_n} B_n$ which extends

$E_{n-1}/\bigcup_{i=1}^{n-1} F_i$, and a chart $f_{n+1}\colon A_{n+1} \times I^q \to M$.

Choose B_n' to be an open neighborhood of $\bigcup_{i=1}^{n} \overline{F}_i$, $\overline{B}_n' \subset B_n$, and A_{n+1}' an open neighborhood of $\overline{F}_{n+1}$, $\overline{A}_{n+1}' \subset A_{n+1}$.

In order to simplify the notation, we will drop the subscripts and write A, A', B, B' E, , f and p for A_{n+1}, A_{n+1}', B_n, B_n', E_n, f_{n+1} and p_n during this induction step.

Let K be a triangulation of $A \cap B$. Then P = $\{\sigma \in K'' : \sigma \cap \bar{A}' \cap \bar{B}' \neq \phi\}$ is a compact polyhedron. If N is a regular neighborhood of P in $A \cap B$ then by J.H.C. Whitehead (see (6), for example), N is a compact manifold with boundary.

We will now trivialize E over N. If U is the image $f(A \times I^q)$ of f, let τ be the composition $\pi_2 f^{-1}\colon U \to I^q$ which assigns a 'level number' to each leaf in U. Let $[-\varepsilon, +\varepsilon]^q$ be an open cube, denoted ε^q, such that $N \times \varepsilon^q \subset (p \times \tau) \circ f(A \times I^q)$. Then the diagram

$$\begin{tikzcd}
E/N \arrow[r, "p \times \tau|"] \arrow[dr, "\tau|"'] & N \times \varepsilon^q \arrow[d, "\pi_2"] \\
& \varepsilon^q
\end{tikzcd}$$

defines the desired trivialization of E over N. Note that $g = (p \times \tau|)^{-1}\colon N \times \varepsilon^q \to E$ is a chart in E which level preserves with f.

We can therefore define a q-isotopy

$$f^{-1} g\colon N \times \varepsilon^q \to A \times \varepsilon^q$$

which is locally trivial since N has codimension 0 in A. Since N is a compact submanifold of A, we can apply the q-isotopy extension theorem to extend this q-isotopy of N in A to an ambient isotopy H of A. The composition $f' = fH|: A'\times \mathcal{E}^q \to M$ is almost the desired new chart map since it matches $E/(\overline{A'}\cap\overline{B'})$.

It remains to restrict f' so that it matches all of $E/\overline{B'}$. Now $E/B'-(A'\cap B')) = E/\overline{B'}-\overline{A'})$ misses $\overline{A'}$. Therefore by compactness we can restrict f' to a small enough set $A'\times\delta^q$ so that $f'(\overline{A'}\times\delta^q)$ meets E only in $E/(\overline{A'}\cap\overline{B'})$ and $\overline{f'(A'\cap\delta^q)}\cap L=\overline{A'}$

Letting $E' = (E/B') \cup f'(A'\times\delta^q)$ it follows that the projection $p'\colon E' \to A'\cup B'$ defined by

$$p'(x) = \begin{cases} p(x) & (x \in E/B') \\ \pi_1 f'^{-1}(x) & (x\in f'(A'\times\delta^q) \end{cases}$$

is well defined.

Recall that B' is a neighborhood of $\bigcup_{i=1}^{n}\overline{F}_i$ and A' is a neighborhood of $\overline{F}_{n+1}$ so that $B_{n+1} = A'\cup B'$ is a neighborhood of $\bigcup_{i=1}^{n+1}\overline{F}_i$ in L. Letting $E_{n+1} = E'$ and $p_{n+1} = p'$, and since $E = E_n$, we now have a microbundle

$$B_{n+1} \to E_{n+1} \xrightarrow{p_{n+1}} B_{n+1}$$

normal to $\mathfrak{F}$ which extends $E_n/\bigcup_{i=1}^{n}F_i$.

We can define a microbundle normal to $\mathfrak{F}$ over the entire leaf L as the limit of the microbundles E_n, restricted to $\bigcup_{i=1}^{n}F_i$.

The structure group of E.

Observe that the structure group of any microbundle normal to a foliation is discrete. For, if (g_1, G_2) and (g_2, G_2) are any charts in such a microbundle then they are also charts in the foliation. Let C be a connected component of $G_1 \cap G_2$. If y is chosen close enough to zero in I^q, there is a $z \in I^q$ such that $g_1(C \times y) = g_2(C \times z)$. Thus $\pi_2 g_2^{-1} g_1(C \times y) = z$. Hence the microbundle has discrete structure group.

We now show that the structure group of E can be chosen at each stage to be contained in $\mathcal{H}(L)$, the holonomy group of L. In the construction of the microbundle E_{n+1}, denote by $f'_{n+1}: A'_{n+1} \times \delta^q \to M$ the new chart which matches E_n. Since E_{n+1} has discrete structure group, f'_{n+1} induces a homeomorphism $\rho_1, \ldots, \rho_k$ of I^q for each component $C_1, \ldots, C_k$ of intersection of A'_{n+1} with charts of B'_{n+1}. Compose f'_{n+1} with (identity $\times \rho_1^{-1}$) so that the composition induces the identity on C_1. For each j, $2 \leq j \leq k$, choose a path in A'_{n+1} from a point x in C_1 to a point y in C_j. Choose a path in $B'_{n+1} - A'_{n+1}$ (which is connected by assumption) from y to x. The composition of these paths is a loop γ based at x, say. The homeomorphism it induces is identity$\circ \mu_n \circ \ldots \circ \mu_1 \circ \rho_j$ by the construction, which is in $\mathcal{H}(L)$ by definition. But $\mu_1, \ldots, \mu_n$ were already chosen to be in $\mathcal{H}(L)$, hence ρ_j is in $\mathcal{H}(L)$. $\square$

<u>Remark</u>. Haefliger has constructed an abstract normal microbundle Ω over all of $\mathfrak{F}$ where the gluing of two charts, $A_1 \times I^q$ and $A_2 \times I^q$, is determined by the holonomy construction. If a single leaf L is proper, then the normal microbundle E obtained from Theorem 1 can be thought of as an embedding in Haefliger's bundle. If L is not proper E can be immersed in Ω.

2.2 Uniqueness of normal microbundles.

Suppose $\mathfrak{F}$ and $\mathfrak{G}$ are codimension p and q topological foliations on an n-manifold M where $p+q \leq n$ and $p, q \geq 1$. Then $\mathfrak{F}$ and $\mathfrak{G}$ are <u>transverse</u> if there is a covering of M by local charts $h: I^{n-p-q} \times I^p \times I^q \to M$, each being a distinguished chart in both $\mathfrak{F}$ and $\mathfrak{G}$, such that $h(I^{n-p-q} \times I^p \times t)$ is a component of a leaf in $\mathfrak{F}$ and $h(I^{n-p-q} \times s \times I^q)$ is a component of a leaf in $\mathfrak{G}$.

<u>Lemma 1</u>. If $\mathfrak{F}$ and $\mathfrak{G}$ are transverse foliations and L is a leaf in $\mathfrak{F}$ there exists a normal microbundle in $\mathfrak{F}$ with fibres contained in the leaves of $\mathfrak{G}$.

<u>Proof</u>. To construct the microbundle we will start with distinguished charts $h_i(I^{n-p-q} \times I^p \times I^q)$ in $\mathfrak{F}$ and $\mathfrak{G}$ covering L, and proceed as in Theorem 1. Recall that in order to make the charts match, it is necessary to compose each chart map with an ambient isotopy H_i. It follows from the fact that these distinguished charts are charts in the foliations $\mathfrak{F}$ and $\mathfrak{G}$ that each ambient isotopy H_i can be chosen to be level-preserving over the levels $I^p \times I^q$. For, if h_1 and h_2 are

two distinguished charts over a leaf L in $\mathcal{F}$ then $\pi_3 h_2^{-1} h_1(I^{n-p-q} \times I^p \times t) = y$, $t, y \in I^q$, and $\pi_2 h_2^{-1} h_1(I^{n-p-q} \times s \times I^q) = x$, $s, x \in I^p$, where defined. Hence after an appropriate reparametrization, $\pi_2 h_2^{-1} h_1(I^{n-p-q} \times (s,t)) = (s,t) \in I^p \times I^q$.

The fibres of the new matching chart $h_2 H_2 \colon I^{n-p-q} \times I^p \times I^q \to M$ will look like $h_2 H_2(r \times I^p \times t) = h_2(r' \times I^p \times t) \subset h_2(I^{n-p-q} \times I^p \times t)$, r and $r' \in I^{n-p-q}$, $t \in I^p$, which is a leaf in $\mathcal{G}$. □

Uniqueness of normal microbundles now follows.

Theorem 2. If E_0 and E_1 are microbundles normal to $\mathcal{F}$ over a proper leaf L, then E_0 and E_1 are isomorphic by a homeomorphism which is isotopic to the identity.

Proof. $E_\varepsilon = (E_0 \times [0,\varepsilon]) \cup (E_1 \times [-\varepsilon,1])$ is a microbundle over $(L \times [0,\varepsilon]) \cup (L \times [-\varepsilon,1])$ in $M \times I$. Recall that in Theorem 1 the extended microbundle is constructed inductively, changing one chart at a time and keeping the previous charts fixed. We can assume local finiteness of the charts in order to construct a normal microbundle E over the leaf $L \times I$ with fibres contained in the leaves of the transverse foliation M,t and keeping E fixed. By standard bundle theory, $(E/(L \times I), E_0)$ is isomorphic to $(E_0 \times I, E_0 \times \{0\})$. The composition $H \colon E_0 \times I \to E \to M \times I$ defines the desired isotopy since H_0 = identity and $H_1(E_0) = E_1$. □

§3. Applications

3.1 Two stability theorems.

As a result of Theorem 1 we first show that the holonomy of a leaf characterizes the foliated neighborhood of the leaf.

Corollary 1. Suppose L_i, $i=1,2$, are proper leaves of $(M_i, \mathcal{F}_i)$. If ψ is a homeomorphism of the leaves L_i such that the diagram (*)

$$\begin{tikzcd} \Pi_1(L_1) \arrow[rr, "\psi_*"] \arrow[dr, "\Phi_1"'] & & \Pi_1(L_2) \arrow[dl, "\Phi_2"] \\ & PL(q) & \end{tikzcd}$$

commutes where Φ_i is the holonomy of L_i, then ψ extends to a homeomorphism ψ' of foliated neighborhoods N_i of L_i.

Proof. Using the construction in the proof of Theorem 1, inductively build normal microbundles N_1 and N_2 with holonomy structure group for the leaves L_1 and L_2 simultaneously. Build N_1 and N_2 from distinguished charts $\{ (g_i, G_i) \}_{i=1}^{\infty}$ in $\mathcal{F}_1$ and $\{ (h_i, H_i) \}_{i=1}^{\infty}$ in $\mathcal{F}_2$, respectively, so that $\psi(G_i)=H_i$.

Recall that at each stage the constructed matching chart (g_i', G_i) is composed with a parameter so that one new component of overlap, C_1, say, induces the identity on I^q. For the leaf L_2 reparametrize the matching chart (h_i', H_i) so that the overlap $\psi(C_1)$ induces the identity also.

The other components $\psi(C_j)$, $j = 2, \ldots, k$. then induce the same element in PL(q) as the corresponding C_j. For we can determine which element C_j induces by choosing a suitable loop γ passing through C_1 and C_j and considering the charts it meets. It is clear that $\psi(C_j)$ determines the same element in PL(q) on examination of the loop $\psi_*(\gamma)$ and observing that the diagram (*) commutes.

We have thus inductively built a homeomorphism between the two microbundles.

We now give two stability theorems.

Corollary 2. Suppose $\mathcal{H}(L) = \{e\}$ where L is a compact leaf of $(M,\mathcal{F})$. Then there is a neighborhood of L in M of the form $L\times D^q$ such that $L\times t$ is a compact leaf for all t in D^q.

Proof. $\mathcal{H}(L)$ is the identity if L is either thought of as a leaf in M or as a leaf in $L\times R^q$. Therefore apply Corollary 1 to obtain homeomorphic microbundles N_1 over L in M and N_2 over L in $L\times R^q$. Let $r = \sup\{\, s\in R$ such that $G_i\times[-s,s]^q\subset N_1$ for all $G_i\subset L\,\}$. Then $L\times[-r,r]^q\subset N_1$ is homeomorphic to a neighborhood of L in $N_2\subset M$. $L\times t$ is a leaf since the homeomorphism preserves leaves.

Corollary 3. Suppose $\mathcal{H}(L)$ is finite and L is compact. Then there is a neighborhood N of L in M such that $N = \bigcup_\alpha L_\alpha$ and $p|L_\alpha \to L$ is a finite covering map where p is a microbundle projection.

In PL and DIFF cases N can be chosen to be the total space of a disc bundle over L.

Proof. Apply Theorem 1 to construct a normal microbundle E over L with charts (g_i,G_i), $i=1,\dots,n$, and with holonomy structure group. Choose a disc D^q so that $g_j(x\times D^q)\subset g_k(x\times I^q)$ for all x in $G_j\cap G_k$. Then choose a conical (compact in TOP) neighborhood C of O in D^q. The map $h|C$ is locally conical at O for all h in $\mathcal{H}(L)$. By finiteness there exists a conical $C_0 \subset C$ such that h is conical on C_0 for all h in $\mathcal{H}(L)$. $B = \bigcap_{h \in \mathcal{H}} h(C_0)$ is a conical neighborhood of O and clearly $h(B) = B$ for all h in $\mathcal{H}(L)$. By standard PL topology, B is a PL ball (see Rourke and Sanderson (6)). (In TOP B is merely a compact set.)

Restrict each chart (g_i, G_i) to $G_i \times B$. Since each overlap map was chosen to be in $\mathcal{H}(L)$, $\pi_2 g_i^{-1} g_k(x \times B) = h_{i,k}(B) = B$, where defined.

Thus for t in B and each chart G_k, where $G_k \cap G_i \neq \phi$, $G_i \times t$ is glued to $G_k \times h_{ik}(t)$. In fact the entire boundary of $G_i \times t$ is glued to other such charts since the boundary of G_i is covered by charts $\{G_k\}$.

Let $L_{i,t} = \{ G_k \times s \subset E|$ such that there exists a sequence of sets $\{ (G_i \times t) = (G_1 \times t_1), (G_2 \times t_2), \dots, (G_n \times t_n) = (G_k \times s) \}$ where $G_i \cap G_{i+1} \neq \phi$ and $t_i = h_{i,i-1}(s_{i-1})$.

Observe that $L_{i,t}$ has no boundary. Since $E|$ is normal in $\mathcal{F}$, $L_{i,t}$ is a leaf of $\mathcal{F}$.

To show that $p|: L_{i,t} \to L$ is a covering map we only have to see that p maps onto L. If x is in L and G_k is a neighborhood of x in L, there exists a sequence of open sets $G_i = G_1, G_2, \dots, G_n = G_k$, say, such that $G_j \cap G_{j+1} \neq \phi$. However $G_i \times t = G_1 \times t$ is glued to $G_2 \times h_{1,2}(t)$ which is glued to $G_3 \times h_{2,3} h_{1,2}(t) \subset G_3 \times B$, and so forth. Therefore $G_n \times h_{n-1,n} \circ h_{n-2,n-1} \circ \dots \circ h_{1,2}(t) \subset G_k \times B \subset L_{i,t}$. The map p is a finite covering map since $\mathcal{H}(L)$ is finite.

In the TOP case the question as to whether N can be a disc bundle over L is unanswered. In the case when $\mathcal{N}(L) = Z_p$ it is equivalent to the following question. If h is a TOP homeomorphism of R^q fixing the origin such that h^p = identity, then does there exist a disc neighborhood D^q of the origin such that $h(D) = D$?

3.2 <u>Submersions and Fibrations</u>.

<u>Definition</u>. A map $f\colon M^n \to W^q$ is a <u>fibration</u> if for every x in W there exists a neighborhood V of x in W and a neighborhood U of $f^{-1}(x)$ in M such that the diagram

$$\begin{tikzcd} f^{-1}(x)\times I^q \arrow[r, "\cong"] \arrow[d, "\Pi_2"'] & U \arrow[d, "f|"] \\ I^q \arrow[r, "\cong"] & V \end{tikzcd}$$

commutes.

We call $f^{-1}(x)$ the <u>fibre</u> of x. The set of all fibres of points in W gives a foliation of M.

<u>Definition</u>. A map $f\colon M^n \to W^q$ is a <u>submersion</u> if for every x in M there exist neighborhoods U of x in M and V of f(x) in W such that the diagram

$$\begin{tikzcd} I^n \arrow[r, "\cong"] \arrow[d, "\Pi"'] & U \arrow[d, "f|U"] \\ I^q \arrow[r, "\cong"] & V \end{tikzcd}$$

commutes. As with a fibration the components of $\{ f^{-1}(x) : x \in W \}$ give a foliation of M.

The following is originally due to Edwards and Kirby in the TOP case.

<u>Corollary 4</u>. A proper submersion $f\colon M^n \to Q^q$ is a fibration.

<u>Proof</u>. We only need show that $L = f^{-1}(x)$, considered as a leaf in the foliation induced by f, has trivial holonomy group.

By Theorem 1, L has a normal microbundle E with discrete structure group. Consider the local charts in E, $g_i\colon G_i \times I^q \to M$, $G_i \subset L$, $i=1,\dots,n$. These charts along with the submersion f give the diagram

$$\begin{tikzcd} G_i \times I^q \arrow[r, "g_i"] \arrow[d, "\pi_2"'] & U_i \arrow[d, "f"] \\ I^q \arrow[r, "j_i"] & W_i \end{tikzcd}$$

where $j_i(t) = f \circ g_i(G_i \times t)$, j_i being clearly well defined.

Let W be a neighborhood of $x = f(L)$ in $\bigcap_{i=1}^{n} W_i$ where $I^q \overset{k}{\cong} W$. Then define a parameter $p_i\colon G_i \times I^q \to G_i \times I^q$ sending (x,t) to $(x, j_i^{-1}k(t))$. Letting $g_i' = g_i \circ p_i$, consider the new diagram

$$\begin{tikzcd} G_i \times I^q \arrow[r, "g_i'"] \arrow[d, "\pi_2"'] & U_i \cap f^{-1}W \arrow[d, "f|"] \\ I^q \arrow[r, "k"] & W \end{tikzcd}$$

It is trivial to check that the diagram commutes and g_i' is a homeomorphism.

The comparison map for any two charts (g_i, G_i) and (g_k, G_k) is therefore the identity since $\pi_2 g_i'^{-1} \cdot g_k'(x,t) = k^{-1} f g_k'(x,t) = \pi_2(x,t) = t$.

Hence E is isomorphic to a microbundle with trivial structure group. From the definition of holonomy group, $\mathcal{H}(L)$ is also trivial.

By Corollary 1 f is a fibration.

In closing we state the Reeb stability theorem.

<u>Theorem.</u> Suppose M is a compact manifold with codimension 1 foliation $\mathcal{F}$. If there is one compact leaf with finite fundamental group, then all leaves are compact and have finite fundamental group.

<u>Proof</u>. In (4) Haefliger gives a proof in the topological case which follows in the PL case using Theorem 1.

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Chapter II.

Unsmoothable Diffeomorphisms

Introduction

The standard terminology of differential topology is used throughout this chapter. In particular, it should be noted that a function possessing continuous derivatives of orders $\leq r$ is of class C^r. A function which possesses continuous derivatives of all orders is of class C^∞ and will sometimes be called 'smooth'. Functions which are topologically conjugate to any smooth function are said to be 'smoothable'.

In 1932 Denjoy [1] constructed a diffeomorphism of the circle which was of class C^1 but which was not smoothable. However he also proved that any diffeomorphism of the circle of class C^r, $r \geq 2$, was smoothable. This naturally leads to the conjecture that every diffeomorphism of class C^2 is smoothable. Indeed, in nearly every other situation in differential topology a C^2 structure implies the existence of a C^∞ structure.

In this chapter we will show that this conjecture is false. In fact we will prove the following:

<u>Theorem</u>. For every $r \geq 0$ and every two-manifold M, there exists a C^r diffeomorphism of M which is not topologically conjugate to any C^{r+1} diffeomorphism.

The basic construction is on the annulus and is easily extended to any two-manifold, hence Denjoy's work cannot be generalized beyond dimension one.

There is an interpretation of these results in terms of foliations. The suspension of a C^r diffeomorphism is a one-dimensional foliation of class C^r which is smoothable if and only if the original diffeomorphism is smoothable. Thus in our case we obtain a one-dimensional C^r foliation of $M \times S^1$ which is not homeomorphic to any C^{r+1} foliation for each $r \geq 0$. Using results from Chapter I, we also show that this foliation is not C^s integrably homotopic to any C^{r+1} foliation, $0 \leq s \leq r$.

I should like to acknowledge the indepent work of C. Fefferman and W. Thurston who have constructed a similar example (unpublished).

§1. Construction of f.

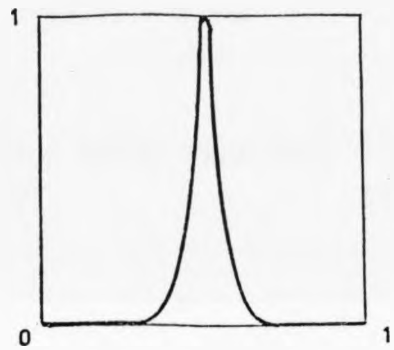
Once and for all choose a smooth 'bump' function

$$\Phi: \quad [0,1] \to [0,1]$$

such that

(i) $\Phi(x) = 0, \ 0 \leq x \leq \frac{1}{3}, \ \frac{2}{3} \leq x \leq 1$ and

(ii) $\Phi(\frac{1}{2}) = 1.$



The nth derivative of Φ, $\Phi^{(n)}$, is clearly not the zero function since Φ is not a polynomial. (Compare Lemma 5 in Section II.)

Let r be a positive integer ≥ 0. Let $\{a_k\}$ and $\{b_k\}$ be positive sequences of real numbers defined by the formulas

$a_k = R/k^{1+\epsilon_r}$, where $\epsilon_r = 1/(r+1)$ and $R = 1/\Sigma_{k=1}^{\infty} (2/k^{1+\epsilon_r})$,

and $b_k = 1/k^{r+1}$. Note the following properties of $\{a_k\}$ and $\{b_k\}$:

(i) $\Sigma \, a_k = \frac{1}{2}$,

(ii) $1/b_k$ is an integer,

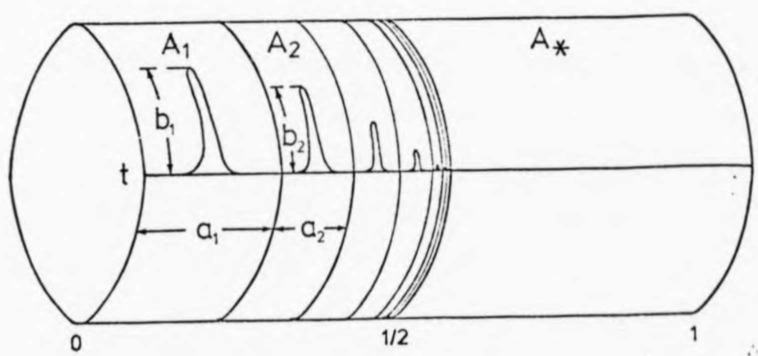
(iii) $\lim_{k\to\infty} b_k/a_k^q = 0$, $0 \leq q \leq r$ and

(iv) $b_k/a_k^{r+1} \to \infty$ as $k \to \infty$.

Define the circle S^1 to be the unit interval $[0,1]$ with 0 and 1 identified and let N denote the annulus $[0,1] \times S^1$. If $p_o = 0$ and $p_k = a_1 + a_2 + \dots + a_k$, let $A_k = [p_{k-1}, p_k] \times S^1$, the subannulus in N of width a_k lying 'between' p_{k-1} and p_k. Denote by A_* the subannulus $\left[\frac{1}{2}, 1\right] \times S^1$.

We can now define a function $f = (f_1, f_2)$ from the annulus N to itself, where $f_1(x,y) = x$ and

$$f_2(x,y) = \begin{cases} y + b_k\Phi((x-p_{k-1})/a_k) & ((x,y) \in A_k) \\ y & ((x,y) \in A_*). \end{cases}$$



The image of $[0,1] \times t$ under f.

Define Π_1 and Π_2 to be the projections onto the first and second coordinates, respectively. It is immediate from the definition of f that all the partial derivatives of $f - id = (f_1 - \Pi_1, f_2 - \Pi_2)$ except for $\partial^n(f_2-\Pi_2)/\partial x^n$ are zero everywhere on the annulus N . The only place where the continuity of $\partial^n(f_2-\Pi_2)/\partial x^n$ is in doubt is on the limit circle $\left[\frac{1}{2} \times S^1\right]$. By the choice of $\{a_k\}$ and $\{b_k\}$,

$$\lim_{k\to\infty} \sup_{x\in\left[p_{k-1},p_k\right]} \left| b_k \partial^n \Phi((x-p_{k-1})/a_k)/\partial x^n \right| =$$

$$\lim_{k\to\infty} \sup_{x\in[0,1]} \left| b_k/a_k^n \, \partial^n \Phi(x)/\partial x^n \right| = \begin{cases} 0 & (0 \leq n \leq r) \\ \text{does not exist} & (n=r+1 \end{cases}$$

so that $\partial^n f_2/\partial x^n$ exists and is continuous if and only if $0 \leq n \leq r$. Therefore f is C^r and is not C^{r+1}.

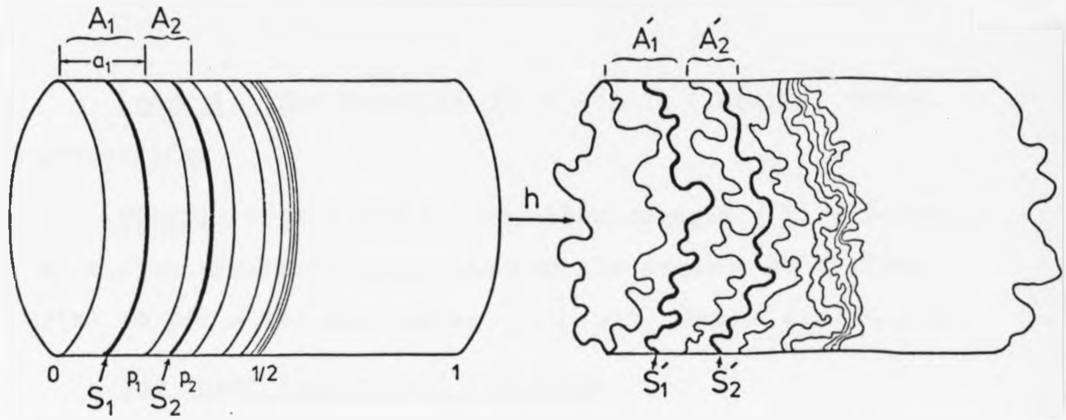
§ 2. Unsmoothability of f.

2.1 Assertion. Let f be the function defined on the annulus N in Section I and extended to the larger annulus $N' = [-1,2] \times S^1$ by the identity. Then there does not exist an embedding $h: N \to N'$ within $\delta < 1/12$ of the identity so that $f' = hfh^{-1}: hN \to hN$ is C^{r+1}.

The somewhat complicated proof of this conjugacy assertion may be clarified by the following remarks:

We are looking for a homeomorphism of the annulus which makes the basic C^r function f on the annulus C^{r+1}. The simplest homeomorphism h which could have a chance of making f smoother is one which changes the relative widths a_k of the subannuli. But no such h can help because from Section I b_k/a_k^{r+1} must tend to 0 if hfh^{-1} is to be of class C^{r+1}. However, from the choice of $b_k = 1/k^{r+1}$, this implies that $\Sigma a_k = \infty$ and the annulus explodes. One might hope to avoid such an outcome by reducing the values of the b_k, the speed at which each subannulus rotates. But since rotation numbers are preserved under a homeomorphism (see Denjoy [1]) this cannot be done effectively. Any shrinking of b_k at one place must be made up by an expansion elsewhere. It would seem that any more complicated homeomorphism would only make the situation worse.

Notation. We will need several lemmas for the proof of the assertion. They will be concerned with the geometry and functional relationships holding in any one subannulus $A_k' = h(A_k)$ with particular regard to the middle 'circle' of the subannulus, $S_k' = h(S_k)$, where $S_k = (p_{k-1} + a_k/2) \times S^1$.



In order to simplify the notation, we will drop the subscript k for the time being and write A, S, A' and S'. We will write b instead of b_k as the maximum distance a point in the subannulus is moved by f.

In the first three lemmas we throw away information about f' restricted to S', leaving a more manageable function which can eventually lead us to a contradiction.

<u>The 'last point' function</u>.

First identify S with S^1. Since h is within δ of the identity, S' has a local ordering inherited from the local ordering of S. (The reader may prefer to work to work with the universal covering spaces $\tilde{S}$ and $\tilde{S}'$ which are actually totally ordered, but we will simplify the notation by working with S and S'.) Thus we can talk about the last point of S' which occurs on any level.

More precisely, a set W is 'on a level x' if $W \subset [0,1] \times x$. In particular, if we define $W_x = S' \cap ([0,1] \times x)$ then W_x is a small set of diameter $< 2\delta$ so there is no ambiguity about order in W_x As the set W_x is closed we can define the 'last point' function $\ell : S \to S'$ by $\ell(x) = \sup(W_x)$. Let L be the image of ℓ.

<u>Lemma 1</u>. The function $\ell : S \to L$ is (locally) order-preserving.

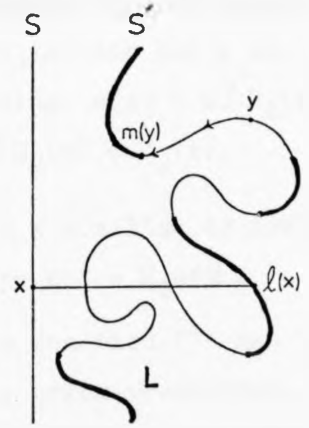
<u>Proof</u>. If $x < y < x + 2\delta$, then clearly $\ell(x) < \ell(x+2\delta)$. By the intermediate value theorem the segment of S' from $\ell(x)$ to $\ell(x + 2\delta)$ must cross $[0,1] \times y$. Hence $\ell(x) < \ell(y)$.

<u>The 'next last point' function</u>.

Define a function m from S' to itself by

$$m(y) = \inf_{\text{in } S'} \{ x \in L : x \geq y \}.$$

We show in Lemma 2(ii) below that $m(y) \in L$. Thus m takes a point in S' and slides it along S' to the 'next last point', that is, the first point in L that it meets.



<u>Lemma 2</u>. The function m satisfies the following properties:

(i) m is weakly order-preserving,

(ii) $m(S') \subset L$ and

(iii) $\Pi_2 m(x) \leq \Pi_2(x)$.

Proof.

(i) Suppose $p < q$ where p and q are in S'. Then $\{ x \in L : x \geq q \} \subseteq \{ x \in L : x \geq p \}$. Hence $\inf \{ x \in L : x \geq q \} \geq \inf \{ x \in L : x \geq p \}$ so that $m(p) \leq m(q)$.

(ii) We show that L is closed from below. Suppose $\{y_i\}$ is a decreasing sequence of points in L which converges to y. Since $\Pi_2|L$ is the inverse of ℓ and hence order-preserving, we have that the $\Pi_2(y_i)$ form a decreasing sequence which clearly converges to $\Pi_2(y)$. Hence $\ell\,\Pi_2(y_i) > \ell\,\Pi_2(y)$ or in other words $y_i > \ell\,\Pi_2(y)$. Taking the limit as $i \to \infty$ it follows that $y \geq \ell\,\Pi_2(y)$. Hence y is in L.

(iii) Since $x \leq \ell\,\Pi_2(x)$ for all x in S' and m is weakly order-preserving, $m(x) \leq m\,\ell\,\Pi_2(x) = \ell\,\Pi_2(x)$. Hence $\Pi_2 m(x) \leq \Pi_2\,\ell\,\Pi_2(x) = \Pi_2(x)$.

We are now in a position to define the desired function f'' from S to itself by $f'' = \Pi_2 m f' \ell$.

Lemma 3. The function f'' has the following properties:

(i) f'' is weakly order-preserving.

(ii) f' carries all points of L vertically at least as far as f'' does. In other words, $f''(\Pi_2 x) - \Pi_2 x \leq \Pi_2 f'(x) - \Pi_2 x$ for all x in L.

(iii) f'' carries every point very close to once around S after $1/b$ steps. That is, $\sum_{n=1}^{1/b} (f''^{n}(x) - f''^{n-1}(x)) \geq 1-2\delta$.

Proof.

(i) It follows from Lemmas 1 and 2(i) that f'' is weakly order-preserving.

(ii). If x is in L, $f''(\Pi_2 x) = \Pi_2 mf' \ell \Pi_2(x) = \Pi_2 mf'(x)$. By Lemma 2(iii), $\Pi_2 mf'(x) \leq \Pi_2 f'(x)$.

(iii) Here we use a weakened version of the fact that rotation numbers are invariant under a homeomorphism to see that the orbit of a point p in S' under f' must have exactly $1/b$ distinct points. It follows that mf' must carry p at least once around S' after $1/b$ steps since m and f' are weakly order-preserving. Observe that if a point y is further along S' than z then the level of y cannot be more than 2δ below the level of z. Hence

$$\sum_{n=1}^{1/b} f''^{n}(x) - f''^{n-1}(x) = \sum_{n=1}^{1/b} \Pi_2(mf')^n(p) - \Pi_2(mf')^{n-1}(p)$$
$$\geq 1 - 2\delta\text{, where } p = \ell(x).$$

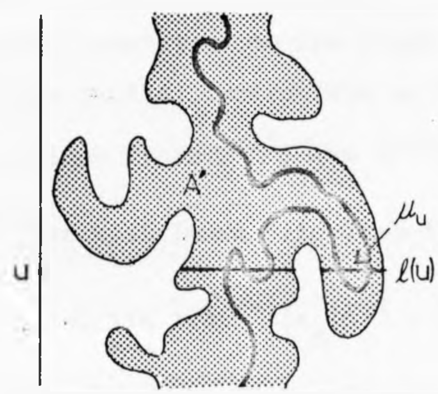
Lemma 4. There is a finite union U of disjoint closed intervals in S of total length $\geq 1/6$ such that $f''(x) - x \geq b/4$ for all x in U.

Proof. Starting with any point q_0 in S, let $q_1, q_2, \ldots q_{1/b}$ be the successive images $q_{n+1} = f''(q_n)$. Let $\{I_j\}$ be the collection consisting of all those intervals $[q_{n-1}, q_n]$ which have length $q_n - q_{n-1} \geq b/2$. Since $\sum_{n=1}^{1/b} (q_n - q_{n-1}) \geq 1-2\delta$ by Lemma 3(iii), it follows easily that the sum of the lengths $|I_j|$ of these longer intervals satisfies $\Sigma|I_j| \geq \frac{1}{2} - 2\delta$. Now let U_j denote the left-hand half of the interval I_j, so that $\Sigma|U_j| \geq \frac{1}{4} - \delta > \frac{1}{6}$.

Since f'' moves the left endpoint of I_j to the right endpoint, and is weakly order-preserving, it follows that $f''(x) - x \geq b/4$ for all x in U_j. □

For each u in U let μ_u denote the connected component of $([-1,2] \times u) \cap A'$ which contains $\ell(u)$. Since S' has strictly positive distance from the boundary of A', it follows that $\inf\{|\mu_u|\} > 0$ so that there is a $\mu \in \{\mu_u\}$ whose length is less than $2\inf\{|\mu_u|\}$. Observe that by Fubini's theorem,

$$\mathrm{Area}(A') = \int_0^1 \mathrm{length}(([-1,2] \times x) \cap A')dx$$
$$\geq \int_U \mathrm{length}(([-1,2] \times u) \cap A')du$$
$$\geq \inf\{|\mu_u|\}/6 > |\mu|/12.$$



<u>Lemma 5</u>. There is a point q in A' such that

$$|\ \partial^n(\Pi_2 f' - \Pi_2)(q)/\partial x^n\ | > b/(4|\mu|^n),$$

$0 \leq n \leq r+1$.

Proof. The line segment μ contains a point q" in the last point set L. By Lemma 3(ii) and Lemma 4, f' must move q" vertically at least the distance b/4. By the construction, f' is the identity on a neighborhood of the boundary of A'. Hence all the partial derivatives up to order r+1 of f'-identity, and therefore $\Pi_2 f' - \Pi_2$, must be zero at the endpoints of μ. It follows from the mean value theorem that there exists a point q' in μ such that $\partial(\Pi_2 f' - \Pi_2)(q')/\partial x > b/4|\mu|$. By repeating this argument n times, there exists a q in μ such that $\partial^n(\Pi_2 f' - \Pi_2)(q)/\partial x^n > b/4|\mu|^n$.

Proof of the assertion. We now consider the differentiable structure of f' on the entire annulus N. When referring to its behaviour on a subannulus A_k, we write q_k, b_k, and μ_k for q, b, and μ.

It follows from the continuity of h that the sequence of points $\{q_k\}$ must tend to the limit 'circle' $h(\frac{1}{2} \times S^1)$ where all the partial derivatives of $\Pi_2 f' - \Pi_2$ are zero. Therefore, since f' is C^{r+1}, $\lim\limits_{k\to\infty} \partial^{r+1}(\Pi_2 f' - \Pi_2)(q_k)/\partial x^{r+1} = 0$. This, together with Lemma 5, implies that $\lim\limits_{k\to\infty} b_k/4|\mu_k|^{r+1} = 0$.

In other words, $\lim\limits_{k\to\infty} 1/4k^{r+1}|\mu_k|^{r+1} = 0$ which implies that $|\mu_k| > 1/k$ for large k. Hence $\Sigma|\mu_k| = \infty$.

On the other hand, since $\mathrm{Area}(A_k') > 1/12|\mu_k|$, we have that $\Sigma|\mu_k| < \infty$

2.2 Generalization to any two-manifold.

Definition. Let M and M' be C^∞ manifolds. A diffeomorphism $f: M \to M$ is topologically conjugate to $g: M' \to M'$ if there exists a homeomorphism $h: M \to M'$ such that $g = hfh^{-1}$.

Theorem. Let M be a C^∞ two-manifold. For each $r \geq 0$ there exists a C^r diffeomorphism F of M which is not topologically conjugate to any C^{r+1} diffeomorphism.

Proof. Choose an annular subset $e(N')$ of M where $e: N' \to M$ is a smooth embedding. Let F be the C^r diffeomorphism of M induced from $f: N' \to N'$ on $e(N')$, and extended to the identity outside of $e(N')$. Now suppose there is a homeomorphism $h: M \to M'$ of C^∞ manifolds such that hFh^{-1} is of class C^{r+1}. It is well known that h can be closely approximated by a C^∞ diffeomorphism $\tilde{h}: M \to M'$ (see J.H.C. Whitehead (9) , Corollary 1.18, or Munkres (7)), hence $(\tilde{h}^{-1}h)F(h^{-1}\tilde{h}): M \to M$ is C^{r+1}. We can therefore assume that h itself is a homeomorphism from M to itself close enough to the identity so that $e^{-1}he|: N \to N'$ is defined and is within 1/12 of the identity. Then the conjugacy $(e^{-1}he)f(e^{-1}h^{-1}e)$ cannot be C^{r+1} by the assertion, but is equal to the C^{r+1} diffeomorphism $e^{-1}(hFh^{-1})e$. □

§3. Application to Foliations.

The suspension of a diffeomorphism. Suppose g is a diffeomorphism of an n-manifold M. The trivial foliation on the product $M \times I$ with leaves $x \times I$ under the identification $(x,0) \sim (gx,1)$ gives rise to a new $(n+1)$-manifold M_g with a one dimensional foliation $\mathfrak{F}_g$ called the suspension of g. The image of $M \times 0$ under this identification will be denoted X. If g is of class C^r, then evidently the manifold M_g and the foliation $\mathfrak{F}_g$ are also of class C^r. Furthermore, if g is C^r-isotopic to the identity, then it is easy to show that M_g is C^r-diffeomorphic to $M \times S^1$. Hence $\mathfrak{F}_g$ gives rise to a C^r foliation of $M \times S^1$.

In this section we will consider under various equivalence relations the special C^r suspension foliation $\mathfrak{F}_F$ obtained from the C^r diffeomorphism F constructed in Section II.

Definition. A foliation $\mathfrak{F}$ on a manifold M is homeomorphic to $\mathfrak{F}'$ on M' if there is a homeomorphism $h: M \to M'$ sending leaves of $\mathfrak{F}$ onto leaves of $\mathfrak{F}'$.

The following technical lemma is similar to Theorem 1.9 of J.H.C. Whitehead (9) and so we only sketch a proof.

Lemma. Let $\mathfrak{F}$ be a one dimensional C^{r+1} foliation on a C^∞ manifold M, $r \geq 0$, and let P be a codimension one topological submanifold of M transverse to $\mathfrak{F}$. Then given any neighborhood U of P there exists a C^∞ submanifold P' of M transverse to $\mathfrak{F}$ and a homeomorphism $\tau: P \to P'$ such that τx lies in the same leaf L and in the same component of $U \cap L$ as x.

Proof. (sketch) P is 'transverse' to $\mathcal{F}$ means that there exist local charts in the foliation homeomorphic to the product $V \times \mathring{I}$ for V an open set in P and $\mathring{I}$ the open unit interval. From these charts choose an open covering E of P contained in U. For each x in P let L_x be the leaf containing x and F_x be the component of $L_x \cap E$ which contains x. By transversality F_x is disjoint from F_y for $x \neq y$ in P. Thus $\{F_x\}$ forms a C^{r+1} foliation of E with C^{r+1} quotient manifold B, homeomorphic to P. Using partitions of unity one can construct a C^{r+1} section $B \to E$. The image of this section can be C^{r+1} approximated by a C^∞ submanifold P' (see [6]) which is also transverse to $\mathcal{F}$. □

Proposition. A suspension foliation $\mathcal{F}_g$ on M_g is homeomorphic to a C^{r+1} foliation if and only if g is topologically conjugate to a C^{r+1} diffeomorphism, $r \geq 0$.

Proof. The product h × identity defines a homeomorphism from $\mathcal{F}_g$ to $\mathcal{F}_{g'}$, where $g' = hgh^{-1}$.

Conversely, let $H: M_g \to M_{g'}$ be a homeomorphism sending $\mathcal{F}_g$ to a C^{r+1} foliation $\mathcal{F}_{g'}$. Observe that the first return map on HX is HgH^{-1}. Let U be a union of local charts in $\mathcal{F}_{g'}$ covering HX. Applying the Lemma, we obtain a homeomorphism τ: HX → X' where X' is a nearby C^∞ submanifold transverse to $\mathcal{F}_{g'}$. Since τ preserves leaves, the C^{r+1} first return map on X' is $\tau HgH^{-1}\tau^{-1}$. □

Corollary 1. Let M be a C^∞ two-manifold. There is a C^r foliation on $M \times S^1$ which is not homeomorphic to any C^{r+1} foliation.

Proof. The example is constructed by suspending F which is C^r-isotopic to the identity.

Definition. Let M be an n-manifold and $\mathcal{G}'$ be the trivial foliation on M×I with leaves M×t. A C^0 foliation $\mathcal{G}$ on M×I is transverse to each level M×t if $\mathcal{G}$ and $\mathcal{G}'$ are transverse foliations (see Chapter 1). (For the definition where $\mathcal{G}$ is a C^r foliation, $r \geq 1$, see (8).)

Two codimension q C^r foliations $\mathcal{F}$ and $\mathcal{F}'$ on M are C^s integrably homotopic, $0 \leq s \leq r$, if there is a codimension q C^s foliation $\mathcal{G}$ on M×I, transverse to each level M×t, and restricting to $\mathcal{F}$ and $\mathcal{F}'$ on the ends.

It is a standard result that if two foliations $\mathcal{F}$ and $\mathcal{F}'$ defined on a compact manifold are C^s integrably homotopic then there is a diffeomorphism of the manifold, isotopic the identity, sending $\mathcal{F}$ to $\mathcal{F}'$, $1 \leq s \leq r$. (See [5] for more details.)

Proposition 2. Suppose $\mathcal{F}$ and $\mathcal{F}'$ are C^0 integrably homotopic on a compact manifold M. Then there is a homeomorphism of M, isotopic to the identity, sending $\mathcal{F}$ to $\mathcal{F}'$.

Proof. We have $\mathcal{G}$, a topological foliation transverse to the trivial 'level' foliation $\mathcal{G}'$ on M×I. Consider a particular leaf M×t in $\mathcal{G}'$. By Lemma 1 in Chapter 1 there exists a normal microbundle over M×t with its fibres contained in leaves of $\mathcal{G}$. Since M×t has trivial holonomy it follows from Corollary 2 in Chapter 1 that this microbundle is homeomorphic to the trivial microbundle by a homeomorphism which (i) preserves levels , i.e. leaves of $\mathcal{G}'$,

(ii) is the identity on M×t and

(iii) 'straightens up' fibres over $x \in$ M×t.

Locally the original microbundle looks like $h(I^{n-p}\times I^p\times I)$ with $h(I^{n-p}\times I^p\times 0) \subset M\times t$ and fibres $h(x\times y\times I)$, $x\in I^{n-p}$, $y\in I^p$. A transverse leaf in $\mathcal{D}$ locally looks like $h(I^{n-p}\times y\times I)$ which is a union of fibres $h(x\times y\times I)$, $x\in I^{n-p}$.

Each fibre is straightened out under the homeomorphism to become $h(x\times y\times 0) \times I$. Thus locally, each leaf $h(I^{n-p}\times y\times I)$ is mapped to $h(I^{n-p}\times y\times 0) \times I$.

By a standard 'piecing-together' argument (see Hudson and Zeeman (3)), we can construct a homeomorphism of all of $M\times I$ sending $\mathcal{D}$ to $\mathcal{F} \times I$.

Thus we have:

Corollary 2. Let M be a compact two-manifold. There is a C^r foliation of $M \times S^1$ not C^s integrably homotopic to any C^{r+1} foliation, $0 \leqslant s \leqslant r$.

Finally we remark that since F is isotopic to the identity, it is easy to show that $\mathcal{F}_F$ is homotopic (in the sense of (4)) to the trivial foliation on $M \times S^1$ with leaves $x \times S^1$.

References for Chapter II.

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